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ON THE DIMENSION OF ATTRACTORS FOR A CLASS OF DISSIPATIVE SYSTEMS*

Iu.S. IL'IASHENKO and A.N. CHETAEV

The paper gives an estimate for the Hausdorff dimension of a global attractive set in a multidimensional problem of pursuit, in terms of the dimension of the pursued manifold. In addition a general result of estimating from above the Hausdorff and entropic dimension of the attractor is obtained for a wide class of dissipative systems.

1. Formulation of the problem. Attractors and the dimensionality. The multidimensional problem of pursuit appears in the theory of large scale biological systems, in particular in constructing a model of a respiratory center /1 – 3/. The problem is motivated in the above articles and is formally stated as follows. Let $\mathbf{R}^N$ be an Euclidean space, $\Lambda \subset \mathbf{R}^N$ is a compact n-dimensional manifold with, perhaps, an edge, and f is $\mathbf{C}^1$-smooth mapping $\mathbf{R}^N \to \Lambda$. The dynamic system

$$x^{\cdot} = f(x) - x \tag{1.1}$$

represents the multidimensional problem of pursuit. The name describes the fact that a point $x(t)$ moving along a phase trajectory is pursuing, at every instant of time, its "shadow" $f(x(t))$ on the manifold Λ. We find that all solutions of the system (1.1) are attracted to some compact set, i.e. to the attractor. The complexity of the "steady state modes" in the system (1.1) is naturally characterized by the dimension of the attractor. The principal result of this paper is represented by the estimate from above obtained for the dimension of the attractor and depending only on n (dimension of the target manifold) and on the Lipshits constants of the mapping f, and independent of the dimension N of the phase space. An analogous result is obtained for a wide class of so-called weakly compressing systems. Such systems include, in spite of the multidimensional problem of pursuit, the Galerkin approximations to various evolution equations, and in particular the Navier—Stokes equations on a two-dimensional torus.

The structure of the attracting sets can be as pathological for the system (1.1) as for the most general dynamic systems. Indeed, let us consider any system $x^{\cdot} = v(x)$, $x \in \mathbf{R}^N$ containing the absorbing sphere $\dot{B}$. The system can be modified outside the sphere B so as to convert it into a system of the form (1.1). To do this we write in the sphere B: $f(x) = v(x) + x$, and let $\Lambda = f(B)$. We extend the mapping f smoothly to the mapping $\mathbf{R}^N \to \Lambda$, and consider the corresponding system (1.1). Within the sphere B this system coincides with the initial system. This means that the pathologies which may be encountered in any dynamic system with an absorbing sphere, will also appear in the multidimensional problem of pursuit.

Thus the attractor of the system (1.1) need not be a manifold or have a dimension in the classical sense. Various definitions of the dimension exist, which can be applied to any (or any compact) subset of the Euclidean space: topological (or inductive introduced by P.S. Uryson), metric (or Hausdorff) and entropic (introduced by L.S. Pontriagin and L.G. Shpirel'man and called by them the "metric order of the compact"). Let us depart for the time being from the exact definitions. From the physical point of view it is desirable to choose a such definition of dimension which would allow, for any compact X belonging to $\mathbf{R}^N$, to obtain an estimate from above, with the dimension of X known, of the number of parameters defining uniquely the position of a point on X. $\mathbf{R}^N$ is regarded here as a phase space, of the physical process and the parameters are its functions on the whole of $\mathbf{R}^N$. This means that the dimension should characterize not only the internal properties of the set X, but also its position in $\mathbf{R}^N$. Let us narrow the problem and consider, as parameters, only the coordinate functions x_i on the space $\mathbf{R}^N$. This yields the following formulation: to find the smallest dimension of the general position plane on which the set X is projected in 1:1 correspondence. The answer should be expressed in terms of the dimension of X and depend on the meaning attached to this dimension. If X is a smooth manifold, then all dimensions listed above coincide with the classical and we have the Whitney lemma /4/, which states that a smooth manifold of dimension n embedded in $\mathbf{R}^N$ projects homeomorphically onto the general position plane of dimension

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$2n+1$. The lemma remains valid when the n-dimensional manifold is replaced by a compact, entropically n-dimensional. The Whitney lemma is clearly false for the topologically n-dimensional compact. For example, the set of all points of the number space $\mathbf{R}^N$ with irrational coordinates only has topological dimension zero, nevertheless it does not project in 1:1 correspondence on any hyperplane. It would be interesting to find out whether the Whitney lemma holds for the Hausdorff n-dimensional sets. We also note that in the course of deriving the numerical estimates for the dimension of the singular attractors (such estimates have appeared lately in greater and greater numbers), it is the entropic dimension which is discussed as a rule. The present paper gives a simultaneous estimate from above for the Hausdorff and entropic dimension of the attractors in weakly compressing systems.

2. Formulation of the results. Definition. The system

$$x^{\cdot} = v(x), \quad x \in \mathbf{R}^N \tag{2.1}$$

in the Euclidean space is weakly compressing if

1°. It has an absorbing region B with compact closure into which all phase trajectories of the system arrive after positive time has elapsed.

2°. $\operatorname{div} v < 0$ in B.

The systems satisfying 1° are often called dissipative. Let the eigenvalues of the quadratic form $(v_*(x)\,\xi,\,\xi)$, $v_* = (\partial v_i/\partial x_j)$, $x \in B$, $\xi \in T_xB$ be equal to $\lambda_1(x) \geqslant \ldots \geqslant \lambda_N(x)$.

Definition. The system (2.1) is called weakly compressing with constants $(\lambda,\, \alpha,\, n)$ (λ is real, α is positive and n is natural), provided that is satisfies the conditions 1° and 2° of the previous definition, and for all x belonging to B $\lambda_1(x) \leqslant \lambda$, $\lambda_{n+1}(x) \leqslant -\alpha$. Sect.3 gives the definitions of the Hausdorff and entropic dimensions, and the fundamental result of this paper is expressed by the following theorem:

Theorem 1. A weakly compressing dynamic system with constants (λ, α, n) has a global attracting set the Hausdorff and entropic dimensions of which do not exceed the quantity $C(\lambda,\, \alpha,\, n)$

$$C(\lambda,\, \alpha,\, n) \leqslant 16n\,(\lambda + \alpha)\,(\lambda + 5\alpha)\,\alpha^{-2} \tag{2.2}$$

$$C(\lambda,\, \alpha,\, n) \leqslant 4n\lambda^2\,(1 + \psi(\lambda))\,\alpha^{-2}$$

where $\psi(\lambda) \to 0$ as $\lambda \to \infty$.

Theorem 2. The system (1.1) in any sphere $B \supset \Lambda$ is weakly compressing, with constants $(L-1,\, 1,\, n)$ where $L = \mathrm{Lip}_B f$.

For the Hausdorff dimension $\dim_H M$ of the attractor M the above estimate can be sharpened; if $\lambda_1(x) + \ldots + \lambda_k(x) < 0$ for all $x \in B$, then $\dim_H M \leqslant k$ for the system (1.1). This yields the estimate $\dim_H M < Ln$ for the system (1.1). A related result but without a quantitative estimate, is obtained in /5/.

Corollary. The system (1.1) has a globally attracting set the Hausdorff and entropic dimensions of which do not exceed the quantity $C(L-1,\, 1,\, n)$. In particular, we have

$$C(L-1,\, 1,\, n) \leqslant 16n\,L(L+4)$$

The multidimensional problem of pursuit is very specific, and we hope that the dimension of the attractor can be estimated in this problem more accurately than for the general, weakly compressing systems.

Hypothesis. The Hausdorff dimension $\dim_H M$ of the attractor M in the multidimensional problem of pursuit is estimated from above by a constant, depending only on dimension n of "the pursued" manifold and independent of the Lipshits constant of the mapping f. It may happen that $\dim_H M \leqslant Kn$ for some $K > 0$.

The students of V.I. Arnol'd have constructed examples of the multidimensional problem of pursuit in which the attractor contains a region of the space $\mathbf{R}^N$, and the dimension of the pursued manifold is $^2/_3\,N$ (D.N. Bernshtein) and $^1/_2 N$ (V.A. Vasil'ev). This implies that in the above hypothesis $K \geqslant 2$. The linear substitution of time $t' = \alpha t$ transforms the weakly compressing system with constants $(\lambda,\, \alpha,\, n)$ into an analogous system with constants $(\lambda/\alpha,\, 1,\, n)$. For this reason we shall only consider the case $\alpha = 1$.

3. Hausdorff and entropic dimensions. Both dimensions are determined for any compact subset of the number space. For the subsets of the k-dimensional manifolds with positive k-dimensional Lebesque measure, both dimensions are equal to k. In the general case they can be expressed by a nonintegral number. Thus the Hausdorff and entropic dimensions of the perfect Kantor set coincide, and are equal to $\ln 2/\ln 3$. In the general case the entropic dimension may be larger than the Hausdorff, and cannot be smaller.

Next we pass to the exact definitions. Let $K \subset \mathbf{R}^N$ be compact. Under the covering compact K we shall understand, in what follows, a finite collection of spheres the union of which contains K. We denote by $\mathbf{U}_\varepsilon(K)$ the class of coverings of compact K consisting of spheres, of radius not greater than ε. Let $U \in \mathbf{U}_\varepsilon(K)$, $U = \{B_\nu\}$, B_ν be a sphere of radius r_ν. We define for any $d > 0$ the d-dimensional volume of covering U as

$$V_d(U) = \sum_\nu r_\nu{}^d$$

Let us fix d and ε, and put

$$m_{\varepsilon, d}(K) = \inf_{U \in \mathbf{U}_\varepsilon(K)} V_d(U)$$

Definition. We call the limit

$$m_d(K) = \lim_{\varepsilon \to \infty} m_{\varepsilon, d}(K)$$

the Hausdorff d-dimensional measure of the set K.

Note. For any compact K the quantity $m_{\varepsilon, d}(K)$, with fixed d, does not decrease monotonuously with decreasing ε. The infimum is taken over even less populated class of coverings. Therefore the measure $m_d(K)$ is defined for any $d > 0$ and can be equal to a positive number, zero or infinity.

Definition. The Hausdorff dimension of the compact subset $K \subset \mathbf{R}^N$ is denoted by $\dim_H K$ and defined as follows. If $m_d(K) = 0$ for all $d > 0$, then $\dim_H K = 0$, otherwise

$$\dim_H K = \sup \{d \mid m_d(K) \neq 0\}$$

The entropic dimension is determined in exactly the same manner as the Hausdorff dimension, except that instead of covering K with arbitrary spheres, we consider the coverings of K consisting of spheres of equal size. The class of all coverings of the compact K by single size spheres of radius not greater than ε, will be denoted by $\mathbf{V}_\varepsilon(K)$. (Every cover of class $\mathbf{V}_\varepsilon(K)$ consists of equal size spheres; different covers of the same class may consist of different spheres). The definition of entropic dimension is obtained by replacing everywhere in the Hausdorff dimension the class $\mathbf{U}_\varepsilon(K)$ by $\mathbf{V}_\varepsilon(K)$. All proofs are carried out below for the Hausdorff dimension only. The proof of Theorem 1 for the entropic dimension is obtained by replacing the class of covers $\mathbf{U}_\varepsilon(K)$ shown above by $\mathbf{V}_\varepsilon(K)$.

4. Volume distortion under the action of the phase flux of a weakly compressing system. Below the k-dimensional volume of the smooth k-dimensional submanifolds $M^k \subset \mathbf{R}^N$ (with or without an edge) is assigned the classical meaning and denoted by $V(M^k)$. If $\Pi^k \subset T_x\mathbf{R}^N$ is a k-dimensional parallelepiped, then $V(\Pi^k)$ represents its k-dimensional volume.

Lemma 1. Let a vector field v satisfy all conditions of Theorem 1 at $\alpha = 1$ and $g: B \to B$ denote the displacement along the phase curves of this field per unit time. Then for any k-dimensional manifolds $M^k \subset B, k > n$ (perhaps with an edge), we have

$$V(gM^k) \leqslant e^{n(\lambda+1)-k} V(M^k) \tag{4.1}$$

and for every $x \in B$

$$\| g_*(x) \| \leqslant e^\lambda \tag{4.2}$$

Proof. Let us fix any $x \in B$ and let $\Pi^k \subset T_x\mathbf{R}^N$ be any k-dimensional parallelepiped. The inequality (4.1) will be proved if we establish that

$$V(g_*(x)\,\Pi^k) \leqslant e^{n(\lambda+1)-k} V(\Pi^k) \tag{4.3}$$

Below we shall neglect the explicit indication of the dependence on x.

Proposition 1. Under the conditions of Theorem 1 we have

$$\begin{aligned} &V(g_*{}^t\Pi^k) \leqslant (1 - t + o(t))^{n-k} (1 + \lambda t + o(t))^n V(\Pi^k) \\ &\| g_*{}^t \| < 1 + \lambda t + o(t) \end{aligned} \tag{4.4}$$

Proof of the proposition 1. Consider a plane P^k stretched over Π^k, and a restriction, of quadratic form (v_*, ξ, ξ) imposed on this plane. Let $\{\eta^1, \ldots, \eta^k\} \subset P^k$ be a normed basis composed of the eigenvectors of this restricting form, μ_j an eigenvalue corresponding to η^j; $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_k$, I^k a cube with the ribs $\eta^1, \ldots, \eta^k$. Clearly, it is sufficient to prove the inequality (4.3) for the particular case of $\Pi^k = I^k$. In this case $V(I^k) = 1$ and $V(g_*{}^t I^k) \leqslant \Pi\, | g_*{}^t \eta^j |$ (the volume of the parallelepiped does not exceed the product of the side lengths). We further have

$$g_*{}^t = E + tv_* + o(t)$$

and hence

$$| g_*{}^t \eta^j | = 1 + t(v_*\eta^j, \eta^j) + o(t) = 1 + t\mu_j + o(t)$$

From the Rayleigh—Courant—Fischer theorem /6/ we have $\mu_j \leqslant \lambda_j$, therefore

$$| g_*{}^t \eta^j | \leqslant 1 + t\lambda + o(t),\ j \leqslant n$$
$$| g_*{}^t \eta^j | \leqslant 1 - t + o(t),\ j > n$$

hence instantly follow inequalities (4.4) and the proposition is proved. Lemma 1 follows at once from the proposition. Let us put $t = 1/\nu$, ν is a natural number. Since $g = (g^{1/\nu})^\nu$ and the volume distortions multiply on superposition, we obtain

$$V(g_*\Pi^k) \leqslant \left(1 - \frac{1}{\nu} + o\left(\frac{1}{\nu}\right)\right)^{\nu(n-k)} \left(1 + \frac{\lambda}{\nu} + o\left(\frac{1}{\nu}\right)\right)^{\nu n} V(\Pi^k)$$

Passing to the limit as $\nu \to \infty$, we obtain the inequality (4.3). The inequality (4.2) is obtained in the same manner from the estimate $(v_*\xi, \xi) \leqslant \lambda(\xi, \xi)$, and this completes the proof of Lemma 1.

Note. The second assertion of Lemma 1 implies that when $\lambda < 0$, then the transformation of the phase flux of the system (2.1), $g: B \to B$ is compressive. In this case the set M consists of a single point, $\dim_H M = 0$ and Theorem 1 holds trivially. Therefore is what follows, $\lambda \geqslant 0$.

5. Estimate of the Hausdorff dimension of the attractor.

Theorem 1 now follows at once from the following lemma.

Lemma 2. Let g be a transformation of the sphere B into itself, with the derivative mapping $g_*(x)$ satisfying the inequalities (4.2) and (4.3) for every $x \in B$. Then the set $M = B \cap gB \cap g^2B \cap \ldots$ is nonempty, compact, and $\dim_H M$ is determined by the inequalities (2.2) by setting in them $\alpha = 1$.

Note. The compactness and nonemptiness of M are obvious. If we assume in addition that M is a smooth manifold, perhaps with an edge, then $\dim M \leqslant n(\lambda + 1)$. Indeed, $gM = M$, therefore $V(gM) = V(M)$. On the other hand, if $d = \dim M$, then by virtue of the inequality (4.1) we have

$$V(gM) \leqslant e^{(\lambda+1)n-d}\, V(M)$$

Therefore $(\lambda + 1)n - d \geqslant 0$, Q.E.D. The proof of Lemma 2 follows from the following two propositions.

Proposition 2. Let the conditions of the lemma hold. Let d be a positive number, $x \in B$ and Q be a unit sphere in T_xB. Let also $R < 1$ exist such that for every $x \in B$ a covering U of the ellipsoid $g_*(x)B$ with spheres of radius R can be constructed, and

$$V_d(U) < R$$

Then $\dim_H M \leqslant d$.

Proof. We put $\delta = 1 - R$ and construct, for every sufficiently small ε and every covering $U \in \mathrm{U}_\varepsilon(M)$, will be constructed a covering $U' \in \mathrm{U}_{2\varepsilon}(M)$ for which

$$V_d(U') < \left(1 - \frac{\delta d}{2}\right) V_d(U) \tag{5.1}$$

From the definition of the Hausdorff measure it follows that in this case $m_d(M) = 0$, which means that $\dim_H M \leqslant d$. First we describe the method of selecting ε. Let $\varepsilon' \in (0, 1)$ be any number for which

$$(1 + \varepsilon')^d (1 - \delta)^d < 1 - \delta d/2$$

The number ε is chosen small enough to ensure that for every $x \in B$ and $h \in \mathbf{R}^N$, with $|h| < \varepsilon$, the following inequality holds (R is the same as in Proposition 2):

$$| g(x + h) - g(x) - g_*(x)h | < \varepsilon' R\, |h| \tag{5.2}$$

Below, every sphere or ellipsoid with center at the point $x \in \mathbf{R}^N$, identifies with a sphere or an ellipsoid of the space $T_x\mathbf{R}^N$ with center at zero, with help of the mapping $\mathbf{R}^N \to T_x\mathbf{R}^N$, $y \mapsto \xi = y - x$. Let now $U \in \mathrm{U}_\varepsilon(M)$, $U = \{B_\nu \mid \nu = 1, \ldots, \varkappa\}$, B_ν be a sphere of radius r_ν with center at x_ν. For every ellipsoid $E_\nu = g_*(x_\nu)\,B_\nu$ (with center $g(x_\nu)$) there exists a covering U^ν with spheres of radius Rr_ν such, that

$$V_d\,(U^\nu) < (1-\delta)\, r_\nu{}^d \tag{5.3}$$

This follows from the conditions of Proposition 2 and the concepts of similarity. We replace now every sphere of the covering U^ν by a concentric sphere of radius $(1+\varepsilon')\,Rr_\nu$ and denote the resulting covering by U'^ν. By virtue of the inequality (5.2), U'^ν is a covering of the compact gB_ν. We denote the totality of all spheres forming the coverings $U'^\nu,\ \nu = 1,\ldots,\varkappa$, by U'. Obviously, U' is the covering of the compact $gM = M$, and it satisfies the inequality (5.1) by virtue of the inequalities (5.2) and (5.3), and this proves Proposition 2.

Proposition 3. Let $A : \mathbf{R}^N \to \mathbf{R}^N$ be a linear operator and let also a nonnegative λ and natural n exist such, that $\| A \| \leqslant e^\lambda$ and for every k-dimensional parallelepiped $\Pi^k \subset \mathbf{R}^N$, $k > n$

$$V\,(A\Pi^k) \leqslant e^{n(\lambda+1)-k}\, V\,(\Pi^k)$$

Then for every d satisfying one of the inequalities

$$d > 16n\,(\lambda+1)(\lambda+5) \tag{5.4}$$
$$d > 4n\lambda^2\,(1+\psi\,(\lambda))$$

there exists a number $R < 1$ and covering U of the ellipsoid AQ by spheres of radius R, for which $V_d\,(U) \leqslant R$. Here Q is a unit sphere in $\mathbf{R}^N$. The function ψ is defined below by the formula (5.6). Here we only note that $\psi\,(\lambda) \to 0$ as $\lambda \to \infty$.

Proof. The covering U is constructed as follows. The ellipsoid AQ is enclosed into the product of two spheres, sphere Q_1 of large radius and small dimension, and sphere Q_2 of small radius and large dimension. First we find the economic covering of the sphere Q_1, then blow up the covering slightly and transform it into the covering of the product $Q_1 \times Q_2$. Let us pass to detailed construction, and take an arbitrary $\rho \in (e^{-1}, 1)$. Let P_1 denote a plane stretched onto all semiaxes of the ellipsoid AQ not smaller than ρ, and P_2 the orthogonal complement to P_1. Let $Q_1 \subset P_1$ be a sphere with center 0 and radius e^λ, and $Q_2 \subset P_2$ a sphere with center 0 and radius ρ.. Clearly, $AQ \subset Q_1 \times Q_2$. We now put $m = \dim P_1$ and estimate m in terms of ρ, λ and n. We take, for every one of the m largest semiaxes of the ellipsoid AQ its preimage of unit length. Let Π^m represent a parallelepiped stretched over these preimages. Clearly, $V\,(\Pi^m) \leqslant 1$. By virtue of the condition and definition we have

$$\rho^m \leqslant V\,(A\Pi^m) \leqslant e^{n(\lambda+1)-m}$$

and hence

$$m \leqslant \frac{n\,(\lambda+1)}{1+\ln\rho}$$

The lemma which follows is easily derived from the Rogers theorems /7/, therefore its proof is not given.

Lemma 3. The m-dimensional sphere of radius $\mathbf{R}$ can be covered by spheres of radius r, the number of which does not exceed the quantity

$$\mathbf{N}\,(m, \mathbf{R}, r) = \frac{(\mathbf{R}+2r)^m}{r^m}\,(\theta\,(m)+1)$$

where $\theta\,(1) = 1$, $\theta\,(m) = m\,(\ln m + \ln\ln m + 5)$ for $m \geqslant 2$.

We choose the number r in such a manner that $\rho^2 + r^2 < 1$ and call all pairs $(\rho, r) : \rho \in (e^{-1}, 1), \rho^2 + r^2 < 1$ admissible. We cover the sphere Q_1 by spheres of radius r the centers of which lie in the plane P_1 and their number estimated using Lemma 3. Next we replace every sphere of the covering by a concentric sphere of radius $R = (\rho^2 + r^2)^{1/2}$ and denote the resulting collection of spheres by $U\,(\rho, r)$. By the Pythagoras theorem, $U\,(\rho, r)$ is a covering of the compact $Q_1 \times Q_2$. We have

$$V_d\,(U\,(\rho, r)) \leqslant \mathbf{N}R^d,\ \mathbf{N} = \mathbf{N}\,(m, e^\lambda, r)$$

therefore

$$V_d\,(U\,(\rho, r)) < R \quad \text{for } d > \frac{\ln \mathbf{N}}{|\ln R|} + 1 \tag{5.5}$$

The right-hand side of the latter inequality is estimated in terms of n, λ, ρ and r, therefore by virtue of the proposition 2 any choice of the admissible pair ρ and r yields an estimate from above for $\dim_H M$. This completes the proof of Theorem 1 in the following, weaker formulation: under the conditions of Theorem 1 there exists an estimate from above for $\dim_H M$, in terms of a quantity depending only on λ/α and n. To prove this weaker theorem we no longer need Lemma 3, and the covering $U\,(\rho, r)$ need no longer be taken as the "almost optimal".

Let us now proceed to the exact estimates. We have

$$\ln N = m\left[\ln(e^{\lambda}+2r) - \ln r + \frac{\ln(\theta(m)+1)}{m}\right]$$

Simple calculations yield

$$\ln(\theta(m)+1) < {}^7/_2 m, \; m = 1, 2, \ldots$$
$$\ln(e^{\lambda}+2r) < \lambda + 1, \; \lambda \geqslant 0, \; r < {}^1/_2$$

therefore for $r < {}^1/_2$ we have

$$\frac{\ln N}{|\ln R|} < \frac{m(\lambda + {}^9/_2 - \ln r)}{|\ln R|} \leqslant \Phi(n, \lambda, \rho, r)$$
$$\Phi(n, \lambda, \rho, r) = \frac{2n(\lambda+1)(\lambda + {}^9/_2 - \ln r)}{(1+\ln\rho)|\ln(\rho^2+r^2)|}$$

Let us construct the covering U for the case when d satisfies the first inequality of (5.4). We put $U = U(\rho, r)$ with $\rho = r = e^{-1/2}$. Clearly,

$$\Phi(n, \lambda, e^{-1/2}, e^{-1/2}) = \frac{4n(\lambda+1)(\lambda+5)}{1-\ln 2} < 16n(\lambda+1)(\lambda+5) - 1$$

By virtue of the estimate (5.5), $V_d(U) < R$, and this proves the first part of the proposition. We define the function ψ by the following equation:

$$\Phi(n, \lambda, e^{-1/2}, [e(\lambda+1)]^{-1/2}) + 1 = 4n\lambda^2(1+\psi(\lambda)) \quad (5.6)$$

Clearly, $\psi(\lambda) \to 0$ as $\lambda \to \infty$. Let d satisfy the second inequality of (5.4) containing this function ψ. We put $U = U(\rho, r)$, with $\rho = e^{-1/2}$, $r = [e(\lambda+1)]^{-1/2}$. Then by virtue of the estimate (5.5) $V_d(U) < R$ and this proves the proposition 3, and together with it Lemma 2 and Theorem 1. We note that with increasing λ it becomes convenient to cover the sphere Q_1 with spheres of small radius r.

6. Proof of Theorem 2.

Let $v(x) = f(x) - x$. Then we have $(v_*(x)\xi, \xi) = (f_*(x)\xi, \xi) - (\xi, \xi) \leqslant (L-1)(\xi, \xi)$, i.e. $\lambda_1(x) \leqslant L-1$. Further, let $K_x = \operatorname{Ker} f_*(x)$. When $\xi \in K_x$, we have

$$(v_*(x)\xi, \xi) = -(\xi, \xi)$$

Since $\operatorname{codim} K_x \leqslant n$, not more than n eigenvalues of the form $(v_*(x)\xi, \xi)$ exceed -1, i.e. $\lambda_{n+1}(x) \leqslant -1$, Q.E.D.

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